Classes 4. MATLAB: Equation systems. Matrix Decomposition

1. Introduction

The Matlab environment can be successfully used to solve equation systems using matrix distributions such as Choleskie decomposition, Gaussian elimination or QR.
Use the chol, lu, or qr command to determine the above matrix distributions. Refer to the Matlab manual for instructions.

2. Choleskie decomposition

Given a symmetric positive definite matrix A, the Cholesky decomposition is an upper triangular matrix R with strictly positive diagonal entries such that[1]

$$A = R' * R,$$

where R' is transpose matrix of matrix A.
Cholesky decomposition is implemented in the Matlab as chol(A). The chol function assumes that A is (complex Hermitian) symmetric. If it is not, chol uses the (complex conjugate) transpose of the upper triangle as the lower triangle. Matrix A must be positive definite[2].

Example 1. Let's use the 6x6 Pascal matrix:

```
>> A=pascal(6)
  A =1 1 1 1 1 1
     1 2 3 4 5 6
     1 3 6 10 15 21
     1 4 10 20 35 56
     1 5 15 35 70 126
     1 6 21 56 126 252
```

This matrix is symmetric and positively defined. Then we determine a triangular matrix

```
>> R=chol(A)

R =

     1 1 1 1 1 1

     0 1 2 3 4 5

     0 0 1 3 6 10

     0 0 0 1 4 10

     0 0 0 0 1 5

     0 0 0 0 0 1:
```

[1]http://mathworld.wolfram.com

[2] https://www.mathworks.com/help/matlab

We can solve the system of equations $A * x = b$. We substitute $A = R' * R$ for the matrix equation and we obtain a new matrix equation

$$R' * R * x = b.$$

Using the matrix left division with the *backslash* operator, we get the solution

```
>> x=R\(R'\b)
```

3. LU decomposition

linalg::factorLU(A) computes an LU-decomposition of an m×n matrix A, i.e., a decomposition of the A into an m×m lower triangular matrix L and an m×n upper triangular matrix U such that $P A = L U$, where P is a permutation matrix[2].

The diagonal entries of the lower triangular matrix L are equal to one (Doolittle-decomposition). The diagonal entries of U are the pivot elements used during the computation. The matrices L and U are unique.

pivindex is a list $[r_1, r_2, ...]$ representing the row exchanges of A in the pivoting steps[2], i.e.,

$B = P\ A = L\ U$, where $b_{ij} = a_{r_i j}$.

Example 2. The system of equations (2) is given, where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -5 & 5 \\ -1 & 10 & 20 \end{bmatrix},\ B = \begin{bmatrix} -3 \\ 5 \\ -5 \end{bmatrix}.$$

We calculate the LU-decomposition by using the command

```
>> [L,U]=lu(A)

L =
      -0.5000  0.0400 1.0000
      1.0000       0      0
      0.5000   1.0000     0

U =
      -2.0000   -5.0000    5.0000
           0    12.5000   17.5000
           0          0 -    0.2000
```

We can quickly determine the solution by using the command

```
>> x=U\(L\b)

x =
      5.0000
      -2.0000
      1.0000
```

Optionally, it is also possible to find the permutation matrix P, see the manual[2].

4. QR decomposition

Given a matrix A, its QR-decomposition is a matrix decomposition of the form

$$A = Q\,R,$$

where R is an upper triangular matrix and Q is an orthogonal matrix, i.e., one satisfying

$$Q^{(T)}\,Q = I,$$

where $Q^{(T)}$ is the transpose of Q and I is the identity matrix. This matrix decomposition can be used to solve linear systems of equations. QR decomposition is implemented in the Matlab as qr(A).

Example 3. For the given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \\ 3 & 3 & 3 \end{bmatrix}$ we find QR-decomposition by using the qr command:

```
A =                 b =
      1  1  1             6
      1  2  3             7
      1  3  6             -4
      3  3  3             19
```

```
>> [Q,R]=qr(A)

Q =
    -0.2887    0.1213      -0.0443      -0.9487
    -0.2887    -0.3638     0.8856       -0.0000
    -0.2887    -0.8489     -0.4428      0.0000
    -0.8660    0.3638 -    0.1328       0.3162

R =
    -3.4641    -4.3301     -5.4848
    0          -2.0616     -4.9720
    0          0           -0.4428
    0          0           0
```

[Q,R] = qr(A,0) produces the economy-size decomposition. If m > n, only the first n columns of Q and the first n rows of R are computed. If m<=n, this is the same as [Q,R] = qr(A).

```
>> [Q,R]=qr(A,0)

Q =
    -0.2887     0.1213     -0.0443
    -0.2887    -0.3638     0.8856
    -0.2887    -0.8489     -0.4428
    -0.8660     0.3638     -0.1328
```

R =

-3.4641	-4.3301	-5.4848
0	-2.0616	-4.9720
0	0	-0.4428

The optional permutation of the columns of decomposition, caused by the presence of the third output argument of the qr command, is useful for detecting peculiarities or decreases in the order of the matrix. A command that displays an additional permutation matrix can be displayed:

```
>> [Q,R,P]=qr(A)
```

and optionally in economy-size version

```
>> [Q,R,P]=qr(A,0)
```

where P is a vector. Continuing the calculation without the permutation matrix[3], the QR decomposition transforms the over-interpreted linear system into an equivalent triangular system. Expression

```
norm(A*x-b)
```

is equivalent[4] to expression

```
norm(Q*R*x-b)
```

Since the multiplication by an orthogonal matrix preserves the Euclidean norm, the above expression is equal to

```
norm(R*x-y)
```

where y = Q '* b. Since the last m-n lines of the matrix R are the same as zero, the expression is split into two parts

```
norm(R(1:n,1:n)*x-y(1:n))
```

and

```
norm(y(n+1:m))
```

If A has a rank of exactly $min\ (m,\ n)$ then it is possible to find a solution x, so the first of the above expressions is equal to zero. The second expression defines the norm of residuum (error).

The solution takes the form

x=R\(Q'∗b)

[3] Note: If you specify a command [Q, R] or [Q, R, P] we get different matrices Q and R.
[4] In Matlab to the accuracy of the error.

with the error specified by

$((n+1){:}m)$

in our case

```
>> y=Q'*b
y =
        -19.0526
        8.4887
        5.1808
        0.3162
>> x=R\(y)
x =
        -6.1000
        24.1000
        -11.7000
```

with an error $y(4{:}4)$= 0.3162

In the case, where matrix A has a rank smaller than $min\ (m, n)$, the triangular structure of the matrix R makes it possible to find a solution by using the least squares method.

5. Exercises

Exercise 5.1. Solve the systems of equations:

a) $\begin{cases} 3x_1 - 2x_2 - x_3 = 5, \\ 2x_1 - 3x_2 - 4x_3 = -5, \\ -x_1 + 2x_2 + x_3 = -1 \end{cases}$

b) $\begin{cases} 6x_1 + 8x_2 + 7x_3 + 3x_4 = 1, \\ 3x_1 + 5x_2 + 4x_3 + x_4 = 2 \end{cases}$

c) $\begin{cases} x_1 - x_2 + 3x_3 = 1, \\ x_1 - 2x_2 - x_3 = 2, \\ 3x_1 - x_2 + 5x_3 = 3, \\ -2x_1 + 2x_2 + 3x_3 = -4. \end{cases}$

What kind of equations are these?

Exercise 5.2. Using the Cholesky method, determine the decomposition $A = L\,L^T$, where

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}.$$

Exercise 5.3. For the given system of equations

a) $\begin{cases} 2x_1 + 4x_2 - 2x_3 + 2x_4 = 0, \\ 4x_1 + 12x_2 + 8x_3 + 4x_4 = 4, \\ -2x_1 + 8x_2 + 40x_3 + 2x_4 = 4, \\ 2x_1 + 4x_2 + 3x_3 + 11x_4 = -18, \end{cases}$

b) $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0, \\ x_1 + 5x_2 + 5x_3 + 5x_4 = -4, \\ x_1 + 5x_2 + 14x_3 + 14x_4 = -4, \\ x_1 + 5x_2 + 14x_3 + 30x_4 = -20 \end{cases}$

find the upper triangular matrix R of the Cholesky decomposition and then find the solution by this method.

Exercise 5.4. Using the Gaussian elimination method for the given matrix A to determine the permutation matrix P and the matrix L and U, where L is a lower triangular matrix, U is a upper triangular matrix and $P\ A = L\ U$.

$$A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & -1 & 0 \end{bmatrix}.$$

Exercise 5.5. Present the matrix A as a product of the upper and lower triangular matrixes. Use this form to solve the system of linear equations (2), where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 11 \\ 14 \\ 8 \end{bmatrix}.$$

Exercise 5.6. Find the QR decomposition (with additional permutation matrix P) for the matrix

$$H = \begin{bmatrix} 1 & 4 & 10 & 20 \\ 1 & 3 & 6 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Exercise 5.7. Using the decomposition QR (full without permutation matrix) find an approximate solution of the system of linear equations (2), where

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \\ 1 & 3 & 2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$